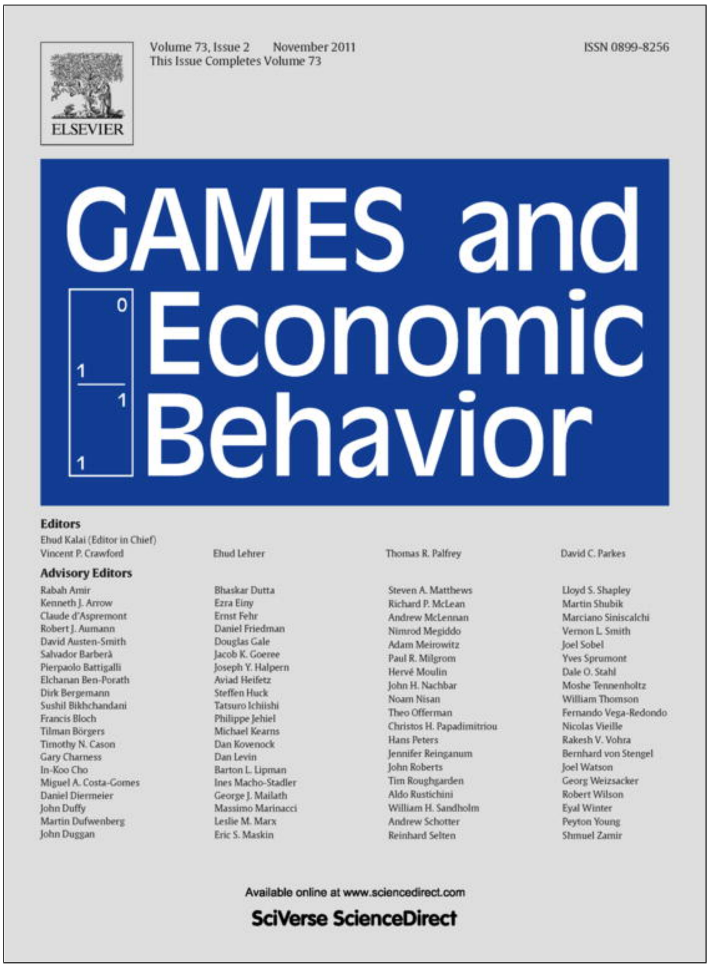

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Top monotonicity: A common root for single peakedness, single crossing and the median voter result

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ABSTRACT

When members of a voting body exhibit single peaked preferences, pair-wise majority voting equilibria (Condorcet winners) always exist. Moreover, they coincide with the median(s) of the voters' most preferred alternatives. This important fact is known as the median voter result. Variants of it also apply when single-peakedness fails, but preferences verify other domain restrictions, such as single-crossing, intermediateness or order restriction. Austen-Smith and Banks (1999) also proved that the result holds under single-peakedness, for a wide class of voting rules that includes the majority rule as a special case, and conveniently redefined versions of a median. We extend and unify previous results. We propose a new domain condition, called top monotonicity, which encompasses all previous domains restrictions, allows for new ones and preserves a version of the median voter result for a large class of voting rules. We also show that top monotonicity arises in interesting economic environments.

1. Introduction

The existence of voting equilibria is crucial in all models of the economy where some of the variables are determined by the political process, rather than set by the market. The tax rates, the level of provision of public goods or the location of public facilities are examples of such variables. Some simple models just try to describe the partial process leading to determine one of these variables. Others incorporate their choice into a larger picture, where markets coexist with the political process and these variables are jointly determined with many others. Public economics, political economy, social choice and other strands of economic analysis base their predictions on the study of equilibria in models of this kind, and yet the existence problems pop out even in the simplest and most stylized versions of reality.

Depending on traditions, authors tend to present their results in different languages, but address essentially the same concerns. In the tradition of social choice, attention focuses on the properties of the social preference relation induced by the voting rule, and the possibility that these social preferences might exhibit cyclical patterns. More positive approaches, like those adopted by political economy, emphasize the question of existence of equilibria, that could be questioned if the core of the game induced by a voting rule was to be empty. In spite of the different language, the concerns in these two approaches are very close, since the dominance condition that is used in discussing the core of a voting game is precisely the social preference relation that social choice theorists worry about. We hope that our exposition will reach readers in both traditions.

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To start with a leading case, remember the possibility of voting cycles (empty cores) under the majority rule and unrestricted preference domains. This is just one instance of the pervasive difficulties that lure behind the use of any type of voting rule. Arrow's impossibility theorem (Arrow, 1970) is a warning that some restrictions are needed to get any form of existence results. Luckily, there are many models of economic interest where individual preferences can be expected to meet the type of requirements that would avoid social preference cycles. A major instance is the case where it can be proven or assumed that voters' preferences are single peaked. Then, pair-wise majority voting leads to well defined equilibrium outcomes. Moreover, these outcomes (called Condorcet winners) are the medians of the distribution of preferred alternatives for the different voters. An analogous result holds for a much wider class of voting rules, as proven by Austen-Smith and Banks (1999).

Single peakedness is the oldest and probably the best known restriction on agents' preferences guaranteeing the existence of voting equilibria (Black, 1958). It is sometimes predicated by assumption, but most often it is derived from assumptions on the basics of simple models.

Much in the same way as convexity induces single peakedness in some reduced models, other general assumptions regarding preferences also induce alternative domain restrictions when applied to simple enough frameworks. This is the case, for example, when preferences satisfy the single crossing property (Mirrlees, 1971; Gans and Smart, 1996, and Milgrom and Shannon, 1994), or the condition of intermediateness (Grandmont, 1978; Rothstein, 1990). Thanks to the implications of these assumptions, it is possible to prove that voting equilibria exist in many models of political economy where some voting rule is to be used. Moreover, these equilibria will be lead to the choice of properly defined medians associated with the voting rule in question.[1]

In this paper we propose a new condition on preference profiles over one-dimensional alternatives, which we call top monotonicity. We prove that top monotonicity can be viewed as the common root of all these classical restrictions, which had been perceived till now as rather different and unrelated to each other. Specifically, we show that single peakedness, as well as the one-dimensional versions of intermediateness and single crossing all imply top monotonicity. In addition, we prove that top monotonicity is sufficient to guarantee the existence of voting equilibria, not only under the majority rule but also for the wide class of voting rules analyzed by Austen-Smith and Banks (1999). Moreover, we show that these equilibria will be closely connected to an extended notion of the median voter.

Therefore, we claim to have achieved several complementary goals. One is to clarify the connections among different restrictions that guarantee the existence of Condorcet winners, by finding their common root. Another is to extend the median-based existence result to preference profiles which allow for much richer combinations of individual preferences than those previously considered, and also to a wide family of voting rules. In fact, classical conditions are encompassed by our restriction, but many other profiles will also pass our test (and thus guarantee existence), while not meeting any of the traditional requirements. A third and nontrivial contribution of the paper is of a more technical nature. Our restriction allows for agents to exhibit indifferences to an extent that classical domain restrictions do not. Given that indifferences on subsets of alternatives may arise in many natural settings, this ability to deal with them may well be considered substantial, in addition to being an obvious technical improvement.

We are aware that a new domain condition, especially when presented in its most general form, raises some immediate questions, that we address in Section 4 and in Appendix A. Just as a preview, here are some, with their suggested answers.

1. When will it be useful to know about this new domain? We claim that top monotonic profiles arise from economic models of interest, like those we discuss in Appendix A, involving the choice between public and private provision of a public good (Epple and Romano, 1996 and Stiglitz, 1974 among others). Authors whose assumptions fit our restriction need no longer resort to ad-hoc arguments and can directly appeal to our results. Moreover, knowledge that this type of restrictions lead to positive results for a great variety of voting rules should encourage authors to weaken their assumptions on preferences, and/or to enlarge their analysis to consider different voting rules.

2. Is the satisfaction of our conditions easy to check? Top monotonicity may be easy to check for in some cases, and also easy to discard, in others. In particular, it requires the preferences of all agents on the alternatives that are tops for some of them to be single plateaued.

We proceed as follows. After this introduction, in Section 2 we present the basic framework to be discussed. In Section 3 we introduce the classical restrictions for the purpose of reference, present our new condition of top monotonicity and prove that it encompasses all the previous ones. In Section 4 we show that the median voter result extends to our new framework, with appropriate qualifications. In Section 5, we discuss why we think that knowing about this new restriction is useful, beyond the obvious fact that it involves a unification and extension of known results. Specifically, we argue that some necessary conditions for our restriction to be satisfied are easy to check, and we also present some stylized economic models where top monotonic profiles arise naturally, while the previously known domain restrictions would not hold. Appendix A works out in some detail two models of tax choice where our condition arises.

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[1] Single crossing and single peakedness turn out to also play an important role for the existence of strategy proof social choice functions (see Moulin, 1980 and Saporiti, 2009 among many others). Our paper does not elaborate on this important incentive compatibility requirement. Notice that single peakedness, single crossing and intermediateness appeared independently of each other in the economics literature, that they do not imply one another, and that each one results from its own underlying logic.

2. The model

It is now time to formalize the different terms that we have used informally in our preceding introduction.

Let A be a set of alternatives and N be a set of agents.

Agents' preferences on the alternatives are complete, reflexive, transitive and continuous binary relations on A. We denote the preference of i by $\succcurlyeq_i$. Its strict part $\succ_i$ is defined so that, for any $x, y \in A$, $x \succ_i y \Leftrightarrow (x \succcurlyeq_i y$ and not $y \succcurlyeq_i x)$. Its indifference part $\sim_i$ is defined so that, for any $x, y \in A$, $x \sim_i y \Leftrightarrow (x \succcurlyeq_i y$ and $y \succcurlyeq_i x)$. The set of all preferences on A is denoted by $\Re$.

Preference profiles are elements of $\Re^n$, and they are denoted by $\succcurlyeq = (\succcurlyeq_1, \ldots, \succcurlyeq_i, \ldots, \succcurlyeq_n)$, $\succcurlyeq' = (\succcurlyeq'_1, \ldots, \succcurlyeq'_i, \ldots, \succcurlyeq'_n)$, etc. For any $a, b \in A$, let $P(a, b; \succcurlyeq) \equiv \{i \in N : a \succ_i b\}$ and $R(a, b; \succcurlyeq) \equiv \{i \in N : a \succcurlyeq_i b\}$.

We now introduce the notions of a preference aggregation rule, voting rule and voting equilibrium. We follow closely Austen-Smith and Banks (1999) since our results will extend those that they present in Chapter 4 of their book.

Definition 1. A preference aggregation rule is a map, $f : \Re^n \to \mathcal{B}$, where $\mathcal{B}$ denotes the set of all reflexive and complete binary relations on A. We denote by $\succcurlyeq_f$ the image of profile $\succcurlyeq$ under aggregation rule f.

Definition 2. Given any two profiles $\succcurlyeq, \succcurlyeq' \in \Re^n$ and $x, y \in A$, an aggregation rule f is:

(1) neutral if and only if $[\forall a, b \in A,\ P(x, y; \succcurlyeq) = P(a, b; \succcurlyeq')$ and $P(y, x; \succcurlyeq) = P(b, a; \succcurlyeq')]$ imply $x \succcurlyeq_f y$ if and only if $a \succcurlyeq_f b$;
(2) monotonic if and only if $[P(x, y; \succcurlyeq) \subseteq P(x, y; \succcurlyeq'),\ R(x, y; \succcurlyeq) \subseteq R(x, y; \succcurlyeq')$ and $x \succ_f y]$ imply $x \succ'_f y$.

A neutral aggregation rule treats all alternatives equally when making pairwise comparisons. Monotonicity implies that if x is socially preferred to y and then some people change their preferences so that the support for x does not decrease, while the support for y does not increase, then x must be still socially preferred at the new profile.

One can always associate to each aggregation rule a family of ordered pairs of coalitions that represent the ability of different groups of agents in determining the social preference relation.

Definition 3. The decisive structure associated with an aggregation rule f, denoted by $\mathcal{D}(f)$, is a family of ordered pairs of coalitions $(S, W) \subseteq N \times N$ such that $(S, W) \in \mathcal{D}(f) \Leftrightarrow S \subseteq W$ and $\forall x, y \in A,\ \forall \succcurlyeq \in \Re^n, [x \succ_i y\ \forall i \in S$ and $x \succcurlyeq_i y\ \forall i \in W \to x \succ_f y]$.

We now notice that we could have started to define our aggregation rule by first providing a family of ordered pairs of coalitions.

Definition 4. A set $\mathcal{D} \subset 2^N \times 2^N$ is

(1) monotonic if $(S, W) \in \mathcal{D}$, $S \subseteq S' \subseteq W'$ and $S \subseteq W \subseteq W'$ imply $(S', W') \in \mathcal{D}$;
(2) proper if $(S, W) \in \mathcal{D}$, $S' \subseteq N \backslash W$ and $W' \subseteq N \backslash S$ imply $(S', W') \notin \mathcal{D}$.

Definition 5. Given a proper set D, the aggregation rule induced by $\mathcal{D}$, denoted $f_{\mathcal{D}}$, is defined as $\forall x, y \in A$, $x \succ_{f_{\mathcal{D}}} y \Leftrightarrow [\exists (S, W) \in \mathcal{D}$: $x \succ_i y\ \forall i \in S$ and $x \succcurlyeq_i y\ \forall i \in W]$.[2]

It is useful to state the connections between preference aggregation rules and decisive structures, because one is closer to the language of social choice and the other is closer to that of public economics and political economy. More precisely, one can ask when it is the case that a decisive structure and a preference aggregation rule can be used interchangeably as being the primitives. This will happen when the decisive structure associated with f induces f again. Austen-Smith and Banks (1999) define voting rules as those aggregation rules that have this property, and provide a characterization for them.

Definition 6. An aggregation rule f is a voting rule if $f = f_{\mathcal{D}(f)}$.

Proposition 1. *A preference aggregation rule is a voting rule iff it is neutral and monotonic.*

In this paper we concentrate on the study of voting rules and their equilibria, which we now define.

Definition 7. Let f be a preference aggregation rule and $\succcurlyeq \in \Re^n$. The core of f at $\succcurlyeq$, $C_f(\succcurlyeq, S)$ is the set of maximal elements in $S \subseteq A$ under the binary relation $\succcurlyeq_f$. Elements in the core of a voting rule will be called voting equilibria.

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[2] Notice that the requirement that $\mathcal{D}$ is proper guarantees that $f_{\mathcal{D}}$ is well defined.

3. Top monotonicity and its relatives

The purpose of this paper is to present a new condition on the domain of definition of voting rules guaranteeing the existence of voting equilibria, or, in other words, the nonemptiness of their core. In this section we introduce this condition, called top monotonicity, and we also discuss its connection with previously known ones.

We begin with some additional notation. For all $i \in N$, for any $S \subset A$, we denote by $t_i(S)$ the set of maximal elements of $\succcurlyeq_i$ on S. That is, $t_i(S) = \{x \in S$ such that $x \succcurlyeq_i y$ for all $y \in S\}$. We call $t_i(S)$ the top of i in S. When $t_i(S)$ is a singleton, $t_i(S)$ will be called i's peak on S.

For each preference profile $\succcurlyeq$, let $A(\succcurlyeq)$ be the family of sets containing A itself, and also all triples of distinct alternatives where each alternative is top on A for some agent $k \in N$ according to $\succcurlyeq$.[3]

Before introducing our new condition, let us recall some classical ones that it will encompass as particular cases. We begin by single peakedness.

Definition 8. A preference profile $\succcurlyeq$ is single peaked iff there exists a linear order $>$ of the set of alternatives such that

(1) Each of the voters' preferences has a unique maximal element $p_i(A)$, called the peak of i, and
(2) For all $i \in N$, for all $p_i(A)$, and for all $y, z \in A$

$$\left[p_i(A) > y > z \text{ or } z > y > p_i(A)\right] \rightarrow y \succ_i z.$$

When convenient, we'll say that a preference profile is single peaked relative to $>$.

Single peakedness requires each agent to have a unique maximal element. Moreover, it must be true for any agent that any alternative z to the right (left) of its peak is preferred to any other alternative that is further to the right (left) of it. In particular, this implies that no agent is indifferent between two alternatives on the same side of its peak. Moreover, indifference classes may consist of at most two alternatives (one to the right and one to the left of the agent's peak).

There are situations where it would be natural to allow for larger indifference classes. Yet, weakening the notion of single peakedness to allow for indifferences is a delicate matter, because it may destroy all regularities in the behavior of voting rules that are guaranteed in single-peaked domains. This is a well known fact, especially in the case of the majority rule (see Exercise 21.D.14 in Mas-Colell et al., 1995), but the distinction between indifferences that do not create cycles and others that do has not been studied systematically. Still, we know that one very important source of breakdown arises when indifferences result from the existence of outside options (see Cantala, 2004). Barberà (2007) describes the complex role of indifferences in domain restrictions.

Among other things, top monotonicity will stretch the extent to which one may accommodate indifferences and still obtain positive results regarding Condorcet winners in the case of the majority rule, or of voting equilibria, more generally. The careful reader will be able to notice at different points that we actually are able to include many combinations of preference profiles where indifferences would preclude the satisfaction of conditions stronger than ours. In particular, we never need to exclude individuals whose preferences are flat over large sets of alternatives. At this point, though, we simply remind the reader of a non-controversial extension that allows for indifferences among top alternatives: it is the idea of single plateaued preferences.

Definition 9. A preference profile $\succcurlyeq$ is single plateaued iff there exists a linear order $>$ of the set of alternatives such that

(1) The set of alternatives in the top of each of the voters is an interval $t_i(A) = [p_i^-(A), p_i^+(A)]$ relative to $>$, called the plateau of i, and
(2) For all $i \in N$, for all $t_i(A)$, and for all $y, z \in A$

$$\left[z < y \leqslant p_i^-(A) \text{ or } z > y \geqslant p_i^+(A)\right] \rightarrow y \succ_i z.$$

When convenient, we'll say that a preference profile is single plateaued relative to $>$.

An important result of Black is that Condorcet winners exist under single peaked preferences, and that they coincide with the median(s) of the distribution of voters' peaks. An elegant extension of the result to the case of single plateaued preferences is due to Fishburn (1973, Theorem 9.3). Another important extension, this time obtained by considering new rules, rather than an extended domain, is due to Austen-Smith and Banks (see Chapter 4, Theorem 4.5 in Austen-Smith and Banks, 1999).

Let us now turn to other types of domain restrictions that have already been proven to be related among them, but are usually considered to be quite separate from the logic of single peakedness. We refer specifically to the one-dimensional versions of single crossing and of intermediate preferences. The latter appears in the social choice literature under the name of order restriction.

[3] This allows for the case where more than one of the three belong to the top of the same agent.

Definition 10. A preference profile $\succcurlyeq$ satisfies the single crossing condition iff there exist a linear order $>$ of the set of alternatives and a linear order $>'$ of the set of agents such that for all $i, j \in N$ such that $j >' i$, and for all $x, y \in A$ such that $y > x$

$$y \succcurlyeq_i x \quad \rightarrow \quad y \succcurlyeq_j x, \quad \text{and}$$
$$y \succ_i x \quad \rightarrow \quad y \succ_j x.$$

When convenient, we'll say that a preference profile is single crossing relative to $>$ and $>'$.

Definition 11. If B and C are sets of integers, let $B \gg C$ mean that every element of B is greater than every element of C. A preference profile $\succcurlyeq$ is order restricted on A iff there is a permutation $\pi : N \to N$ such that for all distinct $x, y \in A$,

$$\{\pi(i)\colon\ x \succ_i y\} \gg \{\pi(i)\colon\ x \sim_i y\} \gg \{\pi(i)\colon\ y \succ_i x\},$$

or

$$\{\pi(i)\colon\ y \succ_i x\} \gg \{\pi(i)\colon\ x \sim_i y\} \gg \{\pi(i)\colon\ x \succ_i y\}.$$

Remark 1. Single crossing and order restriction have been proven to be equivalent (Lemma 2 in Saporiti and Tohmé (2006) and Theorem 3 in Gans and Smart (1996) for the case where the order over the set of alternatives and the order over the set of voters are fixed). We shall use one or the other in our reasonings and comparisons with other conditions, depending on which version is more amenable to treatment in each case.

Both requirements have been frequently used in the political economy literature to prove the existence of Condorcet winners under the majority rule. Indeed, a median result also holds under both preference conditions, since Condorcet winners coincide with the top alternative(s) of the median agent(s) in the order of voters implied by these conditions. As we shall see these results also extend to other voting rules, and to our larger domain.

It is now time to present top monotonicity.

Definition 12. A preference profile $\succcurlyeq$ is top monotonic iff there exists a linear order $>$ of the set of the alternatives, such that

(1) $t_i(A)$ is a finite union of closed intervals for all $i \in N$, and
(2) For all $S \in A(\succcurlyeq)$, for all $i, j \in N$, all $x \in t_i(S)$, all $y \in t_j(S)$, and any $z \in S$

$$[x > y > z \text{ or } z > y > x] \quad \rightarrow \quad \begin{array}{ll} y \succcurlyeq_i z & \text{if } z \in t_i(S) \cup t_j(S) \\ & \text{and} \\ y \succ_i z & \text{if } z \notin t_i(S) \cup t_j(S). \end{array}$$

When convenient, we'll say that a preference profile is top monotonic relative to $>$.

Remark 2. We will say that profile $\succcurlyeq$ is peak monotonic with respect to $>$ if it satisfies the requirements of Definition 11 and, in addition, $t_i(A)$ is a singleton for all i.

We can begin by comparing top monotonicity with single peakedness and single plateauedness to see that it represents a significant weakening of these conditions. Single peakedness requires each agent (i) to have a unique maximal element; and (ii) not to be indifferent between two alternatives on the same side of its peak.

In contrast, top monotonicity allows for individual agents to have nontrivial indifference classes, both in and out of the top. In that respect, it allows for more indifferences than single plateaued preferences do. More importantly, top monotonicity relaxes the requirement imposed on the ranking of two alternatives lying on the same side of the agent's top. Under our preference condition, this requirement is only effective for triples where the alternative that is closest to the top of the agent is itself a top element for some other agent. Moreover, the implication is only in weak terms when the alternative involved in the comparison is top for one or for both agents.

A similar, although less direct comparison can be made between top monotonicity and intermediateness or order restriction. The original conditions involve comparisons between pairs of alternatives, regardless of their positions in the ranking of agents. Top monotonicity is also a strict weakening of these requirements, involving the comparison of only a limited number of pairs (those that are tops for some individuals).

Finally, let us remark that our new definition is predicated on the set of all alternatives, and also on $A(\succcurlyeq)$, i.e., on triples of alternatives which are top for some agent. As we shall see in Section 3, this additional requirement is needed because the property of top monotonicity on a set is not necessarily inherited on its subsets.

We now prove that top monotonicity is a common root for all of the above restrictions, as it is implied by any of them.

Table 1

1	2	3
z	y	x
y	x	y
x	w	z
w	z	w

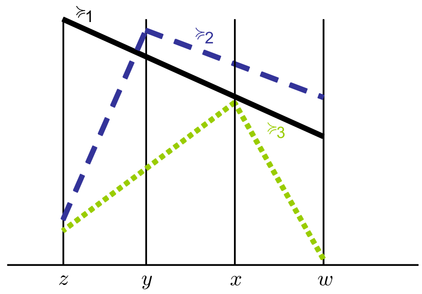

Fig. 1. Preference profile of Example 1.

Theorem 1. *If a preference profile is either single peaked, single plateaued, single crossing or order restricted, then it also satisfies top monotonicity.*

Proof. It is obvious from the definition that single peaked and single plateaued preferences satisfy top monotonicity. Gans and Smart (1996) and Saporiti and Tohmé (2006) prove that single crossing and order restriction are equivalent. Therefore, it will be sufficient to show that single crossing preferences satisfy top monotonicity.[4] This implies showing that top monotonicity holds for the set of all alternatives, and also for each triple of alternatives which are top. Notice that single crossing on all alternatives implies single crossing on triples (as noted by Rothstein, 1990). Therefore, our argument below, which does not appeal to the size of the set of alternatives, covers all cases simultaneously. Let $\succcurlyeq$ be a single crossing preference profile relative to a linear order $>$ of the set of alternatives and to a linear order $>'$ of the set of agents. We now show that $\succcurlyeq$ is top monotonic relative to the linear order $>$ of the set of alternatives. Suppose not. Then, there exist $i, j \in N$, $x \in t_i(S)$, $y \in t_j(S)$, and $z \in S$ such that $y > x$ and $z > y$ (the case in which $x > y$ and $y > z$ is analogous) but it is not the case that (a) $y \succcurlyeq_i z$ if $z \in t_i(S) \cup t_j(S)$ and/or (b) $y \succ_i z$ if $z \notin t_i(S) \cup t_j(S)$. We first consider part (a). Suppose that $z \in t_i(S) \cup t_j(S)$ but $z \succ_i y$. By hypothesis, $x \in t_i(S)$. Therefore, by transitivity, $x \succcurlyeq_i z$ and $z \succ_i y$ imply $x \succ_i y$. Then, if $i >' j$, by single crossing we would have that $y > x$ and $y \succcurlyeq_j x$ imply $y \succcurlyeq_i x$, contradicting the fact that $x \succ_i y$. Second, if $j >' i$, by single crossing we would have that $z > y$, and $z \succ_i y$ imply $z \succ_j y$, contradicting the initial hypothesis that $y \in t_j(S)$. We now consider part (b). Suppose that $z \notin t_i(S) \cup t_j(S)$ but $z \succcurlyeq_i y$. By hypothesis, $x \in t_i(S)$. Therefore, by transitivity, $x \succ_i z$ and $z \succcurlyeq_i y$ imply $x \succ_i y$. Then, if $i >' j$, by single crossing we would have that $y > x$ and $y \succcurlyeq_j x$ imply $y \succcurlyeq_i x$, contradicting that $x \succ_i y$. Second, if $j >' i$, by single crossing we would have that $z > y$, and $z \succcurlyeq_i y$ imply $z \succcurlyeq_j y$, contradicting that $y \in t_j(S)$ and $z \notin t_j(S)$. □

Before we finish the section, let us present some examples and provide some further precisions regarding the requirement of top monotonicity. Examples 1 and 2 show that neither single peakedness implies single crossing nor the converse.

Example 1. Single peakedness without single crossing (similar to Example 2 in Saporiti, 2009). Suppose $A = \{x, y, z, w\}$ and $N = \{1, 2, 3\}$. Assume that preferences are as in Table 1 and Fig. 1. Note that $A(\succcurlyeq) = \{\{x, y, z, w\}, \{x, y, z\}\}$. It is easy to see that the profile is single peaked relative to $w > x > y > z$. However, the profile violates single crossing relative to $w > x > y > z$, for any order $>'$ of the agents (and therefore it is not order restricted either). If $2 <' 3$, $w \succ_2 z$ and $z \succ_3 w$ constitute a violation of single crossing. If $3 <' 2$, $x \succ_3 y$ and $y \succ_2 x$ constitute a violation of single crossing. Similar arguments apply for any other order of alternatives.

Example 2. Single crossing without single peakedness. Suppose $A = \{x, y, z\}$ and $N = \{1, 2, 3\}$. Assume that preferences are as in Table 2 and Fig. 2. Note that $A(\succcurlyeq) = \{\{x, y, z\}\}$. It is easy to see that this preference profile satisfies single crossing on A, relative to $x < y < z$ and $1 <' 2 <' 3$. However, the reader can check that this preference profile is neither single peaked, nor single plateaued.

In Examples 1 and 2, references to single crossing could be changed to order restriction, because the equivalence between both properties. The reader can also check by inspection that, as expected from Theorem 1, top monotonicity is satisfied in both examples.

Example 3 shows that top monotonicity can be satisfied even if none of the previously considered conditions hold.

Example 3. Top monotonicity without single peakedness or single crossing. Suppose $A = \{x, y, z, w\}$ and $N = \{1, 2, 3\}$. Assume that preferences are as in Table 3 and Fig. 3. Note that $A(\succcurlyeq) = \{\{x, y, z, w\}, \{x, z, w\}\}$. It is easy to see that the profile is top monotonic relative to $x < y < z < w$. The preference profile is not single peaked on A because there are triples of

[4] A direct proof that order restriction implies top monotonicity is available from the authors.

Table 2

1	2	3
x	xy	z
y	z	xy
z		

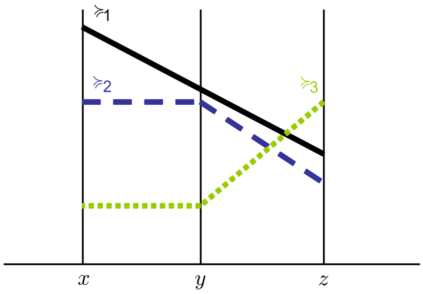

Fig. 2. Preference profile of Example 2.

Table 3

1	2	3
x	z	w
y	w	z
z	y	x
w	x	y

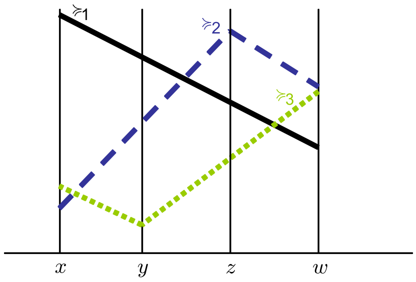

Fig. 3. Preference profile of Example 3.

alternatives such that any of the three are last for the preferences of some agent.[5] Finally, we show that the preference profile is not order restricted for any order of the set of agents, and therefore it is not single crossing. To prove it, consider the three distinct permutations for 1, 2 and 3. If $1 >' 2 >' 3$, $x \succ_1 y$ and $y \succ_2 x$ but $x \succ_3 y$; if $1 >' 3 >' 2$, $z \succ_1 w$ and $w \succ_3 z$ but $z \succ_2 w$; finally, if $2 >' 1 >' 3$, $w \succ_2 y$ and $y \succ_1 w$ but $w \succ_3 y$.

The standard conditions that we have proven to be special cases of top monotonicity share a common feature: when they hold for the universal set of alternatives, they also apply when we restrict attention to any of its subsets, and to triples in particular.[6] In contrast, such inheritance properties from the large to the small do not hold in our case. Example 4 below shows that top monotonicity can be satisfied in a four-alternative profile, and yet not hold when we look at the profile's restriction to a triple. This, of course, extends to larger sets, where we may have top monotonicity at large, and yet not for some subset of alternatives.

Example 4. Let $A = \{x, y, z, w\}$, and $N = \{1, 2, 3\}$. Assume that preferences are as in Table 4. Note that $A(\succcurlyeq) = \{\{x, y, z, w\}\}$. It is easy to see that the profile is peak monotonic (and therefore top monotonic) relative to $x < y < z < w$. However, it violates peak monotonicity on $\{x, y, z\}$ not only relative to $x < y < z$, but also for any other order of the alternatives.

Table 4

1	2	3
x	w	w
y	z	y
w	x	z
z	y	x

What is going on in this example is that there is a cycle between x, y and z, and yet top monotonicity is satisfied in the presence of the fourth alternative w, which is actually the Condorcet winner. Indeed, our domain restriction does not preclude cyclical patterns, but just guarantees that these do not occur at the top of the majority relation. Another way to understand why top monotonicity on a set of alternatives is not inherited on its subsets is by realizing that, as we change the size of relevant sets, we also change the collection of top alternatives, and therefore of the pairs to be compared. This is

[5] This violates a necessary condition for single peakedness when preferences are strict. The condition says that, for any triple of alternatives, one of them cannot be last for any voter. The violation, in our case, appears for all triples.

[6] Defining domain restrictions on triples has a long tradition (see, for example, Sen and Pattanaik, 1969). The part of single peakedness that is actually used in proving the quasitransitivity of the majority relation, and thus the existence of Condorcet winners is precisely the fact that it holds for triples, once it is assumed to hold globally. Notice that the converse statement, ensuring that single peakedness on triples implies single peakedness for the whole preference profile is only true under an additional assumption, involving pairs of agents and 4-tuples of alternatives (see Ballester and Haeringer, 2007).

why, in order to control some limited relationships among some triples of alternatives, top monotonicity is required to hold not only on the universal set A, but also on triples of alternatives which are top for some agents.

One last interesting point regarding our new definition is that, when we restrict attention to preference profiles defined on triples, and where no agent is fully indifferent, then top monotonicity is equivalent to single crossing.[7] This shows that it is important to preserve the level of generality that we have achieved in our definition, if we want to have a domain restriction which really supersedes all those that we have considered as predecessors. In particular, the encounter of single peakedness and single crossing could not have been reached if we had insisted on looking at triples as the starting point of our analysis.

4. Weak medians of the tops and the existence of Condorcet winners

In this section, we show that top monotonicity guarantees the existence of voting equilibria under any voting rule, and that these will be closely connected to an extended notion of the median voter. Before stating this second result of the paper, we introduce some additional notation, and we propose an extension of the notion of median.

Let $>$ be a linear order of the set of alternatives and $\succcurlyeq$ be a preference profile. For any $z \in A$, we define the following three sets

$$N_{\{z\}} = \{j \in N\colon\ z \in t_j(A)\},$$

$$N_{\{z\}^-} = \{k \in N\colon\ z > x \text{ for all } x \in t_k(A)\},$$

and

$$N_{\{z\}^+} = \{h \in N\colon\ z < x \text{ for all } x \in t_h(A)\}.$$

We remark that when $\succcurlyeq$ is top monotonic relative to $>$, and z is in the top of some agent i, then $N_{\{z\}} \neq \emptyset$ and the three sets $(N_{\{z\}^-}, N_{\{z\}}, N_{\{z\}^+})$ constitute a partition of the set of voters N. Indeed, $N_{\{z\}}$ contains all voters, including i, for whom z is in the top. $N_{\{z\}^-}$ (resp. $N_{\{z\}^+}$) contains all voters for which all top elements are to the left (resp. to the right) of z. Clearly, then, these three sets are disjoint. To prove that their union contains all elements of N, suppose not. For some agent l, z should not be in l's top, while some alternatives x and y, one to the right and one to the left of z, should belong to the top of l. But then, by top monotonicity we would have $z \succcurlyeq_l x$ and also $z \succcurlyeq_l y$. Since x and y are both top for l, so is z, a contradiction.[8]

Let n, $n_{\{z\}}$, $n_{\{z\}^-}$, and $n_{\{z\}^+}$ be the cardinalities of N, $N_{\{z\}^-}$, $N_{\{z\}}$ and $N_{\{z\}^+}$, respectively. From the remark above, we know that if z is in the top of some agent, then $n_{\{z\}} + n_{\{z\}^-} + n_{\{z\}^+} = n$. The following definition will allow us to establish an analogue of the classical median voter result for the case of top monotonic profiles.

Definition 13. Let f be a voting rule. An alternative z is a weak f-median top alternative in a top monotonic profile $\succcurlyeq$ relative to an order $>$ of the set of alternatives iff

(1) z is a top alternative in $\succcurlyeq$ for some agent, and
(2) $(N_{\{z\}^-}, N_{\{z\}^-} \cup N_{\{z\}}) \notin \mathcal{D}(f)$ and $(N_{\{z\}^+}, N_{\{z\}} \cup N_{\{z\}^+}) \notin \mathcal{D}(f)$.[9]

We will denote by $WM_f(\succcurlyeq)$ the set of weak f-median top alternatives at that profile. We define m^- and m^+ as the lowest and the highest elements in this set according to the order $>$ at that profile.

Definition 14. An alternative z is an extended weak f-median in a top monotonic profile $\succcurlyeq$ relative to an order $>$ of the set of alternatives iff $m^- \leqslant z \leqslant m^+$.

It is not hard to prove that extended medians in our sense always exist. We will denote by $M_f(\succcurlyeq)$ the set of extended weak f-median alternatives at that profile.

We can now state and prove the following result.

[7] A formal proposition and a proof of that fact is available from the authors.

[8] Notice that our definition of top monotonicity does not preclude the possibility that an agent's top might be non-connected relative to the order of $>$. Informally, what it demands is that, if an agent has two peaks with a valley in between, then no other agent's peak lies in that valley. In that sense also, our condition is less demanding than that of single plateaued, where we assumed that the tops are connected.

[9] When f is the majority rule we say that an alternative z is a weak median top alternative in a top monotonic profile relative to an order $>$ of the set of alternatives iff (1) z is a top alternative in for some agent, and (2) $n_{\{z\}^-} + n_{\{z\}} \geqslant n_{\{z\}^+}$ and $n_{\{z\}} + n_{\{z\}^+} \geqslant n_{\{z\}^-}$.

Theorem 2.

(1) *Let f be a voting rule. Whenever a profile of preferences $\succcurlyeq$ is top monotonic relative to some order $>$, $C_f(\succcurlyeq)$ is not empty and $C_f(\succcurlyeq) \subseteq M_f(\succcurlyeq)$.*
(2) *If the profile of preferences $\succcurlyeq$ is peak monotonic, $WM_f(\succcurlyeq) \subseteq C_f(\succcurlyeq)$.*

Proof. Statement (2) is an immediate corollary of (1) when all agents' tops are singletons. We now prove statement (1). Let the preference profile $\succcurlyeq \in \Re^n$ be top monotonic on A relative to $>$ and let f be a voting rule. The strategy of proof involves showing that

(a) $C_f(\succcurlyeq, WM_f(\succcurlyeq))$ is not empty. To establish that, we show that $f(\succcurlyeq)$ is quasitransitive on $F(\succcurlyeq)$, where $F(\succcurlyeq)$ is the set of alternatives that belong to the top set of some agent.[10] As we shall see, the argument uses the part in the definition of top monotonicity which requires the property to hold on triples of alternatives that are top for some agent.
(b) Any alternative $x \in WM_f(\succcurlyeq)$ is such that $x \succcurlyeq y$ for any $y \in M_f(\succcurlyeq)\backslash WM_f(\succcurlyeq)$.
(c) Any alternative $y \notin M_f(\succcurlyeq)$ is such that $x \succcurlyeq y$ by some element $x \in WM_f(\succcurlyeq)$, and
(d) Any alternative $x \in C_f(\succcurlyeq, WM_f(\succcurlyeq))$ is such that $x \succcurlyeq y$ for any $y \notin M(\succcurlyeq)$. Steps (a), (b) and (d) imply that the elements in $C_f(\succcurlyeq, WM_f(\succcurlyeq))$ that we identify in (a) belongs to $C_f(\succcurlyeq, A)$ and (c) proves that no alternative outside $M(\succcurlyeq)$ belongs to $C_f(\succcurlyeq, A)$. Notice that conclusion (b) does not preclude the possibility of additional elements in $M(\succcurlyeq)$ but not in $WM(\succcurlyeq)$ also belonging to $C_f(\succcurlyeq, A)$.

(a) To prove that $C_f(\succcurlyeq, WM_f(\succcurlyeq))$ is not empty, it is enough to show that the $\succcurlyeq_f$ is quasitransitive on $F(\succcurlyeq)$. Let, w.l.o.g., $x, y, z \in F(\succcurlyeq)$ be such that $x < y < z$. Top monotonicity, by definition, holds on each such triple, since each of the elements in $F(\succcurlyeq)$ is top for some agent. First notice that if one of the admissible preference relations in the profile has y as its unique top alternative, then top monotonicity requires that all preferences in this profile should be single plateaued. In that case, it is well known that the $\succcurlyeq_f$ is quasitransitive.

Also notice that, since $\succcurlyeq$ is top monotonic relative to $x < y < z$, preferences where $x \succ_i z \succ_i y$, $z \succ_i x \succ_i y$ and $x \sim_i z \succ_i y$ cannot be part of the profile. In view of the preceding remarks, we are left with the cases where our preference profile is a combination of the preferences that appear below:

$\succcurlyeq_1$	$\succcurlyeq_2$	$\succcurlyeq_3$	$\succcurlyeq_4$	$\succcurlyeq_5$	$\succcurlyeq_6$
x	xy	x	z	zy	z
y	z	yz	y	x	xy
z			x		

We write $a \succ_f b$ iff $[\exists(S, W) \in \mathcal{D}(f): \forall i \in S: a \succ_i b$ and $\forall i \in W: a \succcurlyeq_i b]$ for all $a \neq b$, $a, b \in \{x, y, z\}$. Therefore, if $a \succ_f b$ we say that $(P(a, b; \succcurlyeq), R(a, b; \succcurlyeq)) \in \mathcal{D}(f)$ where $P(a, b; \succcurlyeq) \equiv \{i \in N\colon\ a \succ_i b\}$ and $R(a, b; \succcurlyeq) \equiv \{i \in N\colon\ a \succcurlyeq_i b\}$.

We must prove that: $x \succ_f y$ and $y \succ_f z$ implies $x \succ_f z$, $x \succ_f z$ and $z \succ_f y$ implies $x \succ_f y$, $y \succ_f z$ and $z \succ_f x$ implies $y \succ_f x$, $y \succ_f x$ and $x \succ_f z$ implies $y \succ_f z$, $z \succ_f x$ and $x \succ_f y$ implies $z \succ_f y$ and $z \succ_f y$ and $y \succ_f x$ implies $z \succ_f x$.

We provide the argument for the case $x \succ_f y$ and $y \succ_f z$, proving that this implies $x \succ_f z$. Other proofs are left to the reader. Since $x \succ_f y$, $(P(x, y; \succcurlyeq), R(x, y; \succcurlyeq)) \in \mathcal{D}(f)$, and since $y \succ_f z$, $(P(y, z; \succcurlyeq), R(y, z; \succcurlyeq)) \in \mathcal{D}(f)$.

We must show that $(P(x, z; \succcurlyeq), R(x, z; \succcurlyeq)) \in \mathcal{D}(f)$.

In fact, not all combinations of these preferences are compatible, given that our profile satisfies top monotonicity. Specifically, $\succcurlyeq_2$ and $\succcurlyeq_3$ or $\succcurlyeq_5$ and $\succcurlyeq_6$ are not mutually compatible. Therefore, either $P(x, y; \succcurlyeq) \subseteq P(x, z; \succcurlyeq) \subseteq R(x, z; \succcurlyeq)$ and $P(x, y; \succcurlyeq) \subseteq R(x, y; \succcurlyeq) \subseteq R(x, z; \succcurlyeq)$ or $P(y, z; \succcurlyeq) \subseteq P(x, z; \succcurlyeq) \subseteq R(x, z; \succcurlyeq)$ and $P(x, z; \succcurlyeq) \subseteq R(x, z; \succcurlyeq) \subseteq R(y, z; \succcurlyeq)$. Since $\mathcal{D}(f)$ and f is neutral, we have that $(P(x, z; \succcurlyeq), R(x, z; \succcurlyeq)) \in \mathcal{D}(f)$. Therefore, $x \succ_f z$.

(b) We now show that no $y \in WM_f(\succcurlyeq)$ loses against any alternative $x \in M_f(\succcurlyeq)\backslash WM_f(\succcurlyeq)$. Suppose that $x > y$ (the case $y > x$ is identical). Because x is not a top alternative, by top monotonicity $y \succ_i x$ for all $i \in N_{\{y\}^-} \cup N_{\{y\}}$. Since y is a weak f-median top alternative, $(N_{\{y\}^+}, N_{\{y\}} \cup N_{\{y\}^+}) \notin \mathcal{D}(f)$. Therefore $y \succcurlyeq_f x$.

(c) We'll show that $m^- \succ_f x$ for any alternative x to its left, and that $m^+ \succ_f x$ for any alternative x to its right. We provide the argument for m^- and $x < m^-$. Notice that, since m^- is the lowest weak f-median top alternative $(N_{\{m^-\}^-}, N_{\{m^-\}} \cup N_{\{m^-\}^-}) \notin \mathcal{D}(f)$. By top monotonicity, notice that,

$$\left\{i \in N\colon\ x \succ_i m^-\right\} \subseteq N_{\{m^-\}^-},$$

[10] A binary relation is quasitransitive if its strict part is transitive. This condition is stronger than acyclicity and like transitivity it is enough to check that it holds on any triple. It is obviously sufficient to guarantee the existence of maximal elements for the relation on any finite subset of A. Therefore, the case for A finite does not require further comment. When A is an infinite set, we have made enough assumptions as to guarantee the existence of maximal elements as well, for any quasitransitive relation. These assumptions are (1) the fact that tops are compact, and (2) the continuity of preferences. The interested reader is referred to the literature on the topic of existence of maximal elements, in particular to Kalai and Schmeidler (1977) and Walker (1977). A more detailed discussion of the issue is available upon request.

that

$$\{i \in N:\ x \sim_i m^- \text{ and } m^- \in t_i(S)\} \cup$$
$$\{i \in N:\ m^- \succ_i x \text{ and } m^- \in t_i(S)\} = N_{\{m^-\}},$$

and that

$$\{i \in N:\ m^- \succ_i x \text{ and } m^- < t_i(S)\} = N_{\{m^-\}^+}.$$

Hence, $P(x, m^-; \succcurlyeq) \subseteq N_{\{m^-\}^-}$, $R(x, m^-; \succcurlyeq) \subset N_{\{m^-\}} \cup N_{\{m^-\}^-}$. Moreover, $N_{\{m^-\}^+} \subset P(m^-, x; \succcurlyeq)$ and $R(m^-, x; \succcurlyeq) \equiv N_{\{m^-\}} \cup N_{\{m^-\}^+}$. Therefore, $m^- \succ_f x$ for any $x < m^-$.

(d) Finally, it is not hard to prove that if an alternative w in $WM_f(\succcurlyeq)$ is such that $w \succcurlyeq x$ for any $x \in WM(\succcurlyeq)$, then $w \succcurlyeq_f m^-$ and $w \succcurlyeq_f m^+$, and therefore $w \succcurlyeq_f y$ for any $y \notin M(\succcurlyeq)$. □

5. How useful is our new restriction?

In Section 2 we have proven that top monotonicity is a weakening of classical domain restrictions. This gain in generality is clarifying, since it exhibits the common root of conditions that have been till now perceived as quite unrelated and that are indeed independent from each other, as shown by Examples 1 and 2. In Section 4 we have shown that this gain in scope still allows for an existence result of Condorcet winners where medians play a central role. In the present section, we want to address the following two questions: (1) When confronted with a given preference profile, can we easily recognize whether it satisfies top monotonicity? (2) Are there interesting economic models where top monotonicity holds, while previously known conditions do not?

5.1. Necessary conditions for top monotonicity

To answer the first question, we present a condition that is easy to check and that is necessary for a profile to be top monotonic. A simple version of this result applies for peak monotonic profiles. In this case, where each agent has a singleton top, the condition requires that the profile of preferences over the peaks of individuals be single peaked. This is easy to check. Establishing peak monotonicity may be harder, but discarding it is a simple matter. In the general case of top monotonicity, we can establish a similar necessary condition, which is close to requiring that preferences of tops should be single plateaued. Proposition 2, whose proof we leave to the reader, will make this intuition more precise.

Given a preference profile let $F(\succcurlyeq)$ be the set of alternatives that belong to the top set of some agent.

Definition 15. A preference profile $\succcurlyeq$ is weakly single plateaued on $S \subset A$ relative to a linear order $>$ of the set of alternatives, iff

(1) Each of the voters' preferences has a unique maximal interval $t_i(S) = [p_i^-(S), p_i^+(S)]$, called the plateau of i, and
(2) For all $i \in N$, for all $t_i(S)$, and for all $y, z \in S$

$$[p_i^-(S) \geqslant^{11} y > z \text{ or } z > y \geqslant p_i^+(S)] \to y \succcurlyeq_i z.$$

Proposition 2. *A preference profile $\succcurlyeq$ is top monotonic on A relative to a linear order $>$ of the set of alternatives only if it is weakly single plateaued on $F(\succcurlyeq)$ relative to the same linear order when restricted to the set of alternatives in $F(\succcurlyeq)$.*

In the case of peak monotonic profiles, weak single plateauedness on $F(\succcurlyeq)$ collapses to the standard condition of single peakedness on $F(\succcurlyeq)$. Therefore, in that case the result stated in Proposition 2 resembles Lemma 4 in Saporiti and Tohmé (2006), which shows that a single crossing profile, once restricted to the set of ideal points, always satisfies single peakedness.

5.2. Economic models giving rise to top monotonicity

Top monotonicity leaves room for new types of preferences that arise from the analysis of economic models.

Consider for example an agent who can guarantee herself the maximum of two utilities on an interval $[0, T]$, as illustrated in Fig. 4.

Then, the attainable utilities by her choices on $[0, T]$ are represented by the upper envelope of two curves. This agent will have two local peaks, one which (at least) will be global.

Assume that, in addition to this general structure, the specific shapes of the preferences of different agents are such that:

(a) There exist two points B and B', $B < B'$, such that the left global peaks of the agents will be attained below B, and the right global peaks will be attained above B', and

[11] By $\geqslant$ we mean that either $x = y$ or else $x > y$.

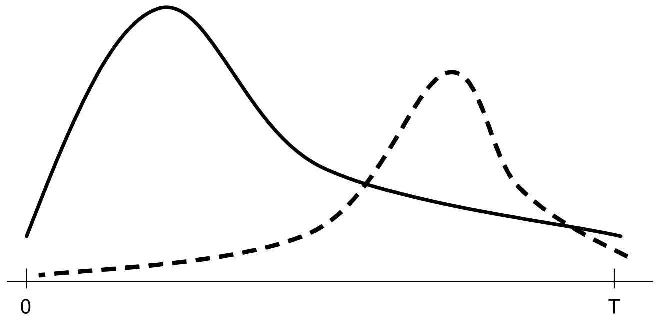

Fig. 4. Two utilities whose envelope represents the preference of an agent.

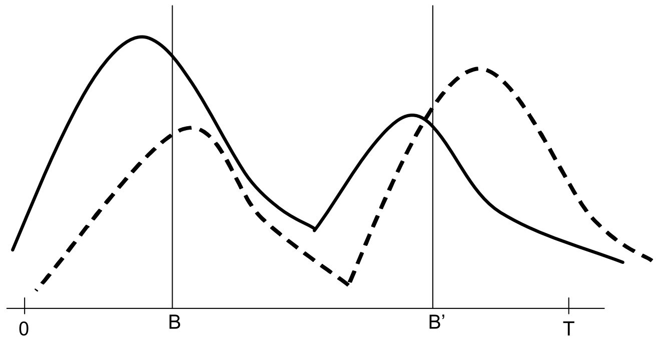

Fig. 5. General structure of top monotonicity.

(b) If the global peak of an agent i is below B this agent prefers B' to any alternative above B' and if the global peak of an agent i is above B' this agent prefers B to any alternative below B.

The reader can check that profiles of these double peaked preferences, as shown in Fig. 5, arising from such a construction do satisfy top monotonicity. We propose this particular structure because it captures the main features of profiles that arise when solving for the preferences of individuals in models of public economics where two modes of provision of a service are possible. Example of these include the choice of a tax rate to finance a public good (Stiglitz, 1974) or the choice of a tax rate to finance public schooling in the presence of an option to buy private schooling (Epple and Romano, 1996).[12]

In both cases, one of the maxima is attained at 0 (which plays the role of our B point), and the other at some point beyond that which would make the individual indifferent between the public and the private option (this is our point B' above). The additional connections between the preferences of different agents which determine whether or not top monotonicity is satisfied, depend on well defined economic variables.

Admittedly, top monotonicity is not always equally useful. In the models where it is, there will exist regions where no global peak is to be found. This is the role of B and B'. We are aware that, in some applications, it is useful to assume that all alternatives are the unique peak for some agent. When this is the right modeling decision, we have little new to offer, since then top monotonicity collapses to single peakedness (and so do all other classical domain conditions). Similarly, if any subset of the set of alternatives is a plateau for some agent, top monotonicity collapses to single plateaued preferences.

Even in these cases, where preferences domains are assumed to be so rich, we have something to contribute. Our previous analysis shows that, if one is ready to work under the assumption that any subset is a top for someone, then all other classical conditions collapse to that of single peakedness. This gives special value to that classical condition.

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[12] A detailed analysis of the models in these papers and the assumptions under which top monotonicity holds is available from the authors.

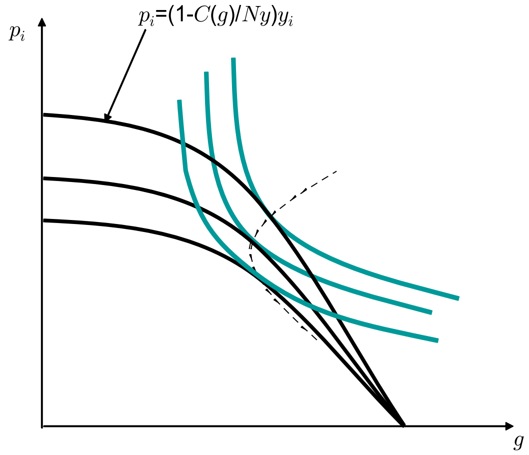

Fig. A.1. Demand for the public good as a function of y_i with a convex technology.

Appendix A

We briefly describe below two models of tax rate determination that induce top monotonicity.

A.1. Voting for a level of expenditure

The first model is based on Stiglitz (1974). There are two goods, a public good, g and a private good, p. The set of voters and of taxpayers are identical. Wealth (given exogenously) of the ith individual is denoted by y_i. Thus total national wealth is just $\sum y_i = \bar{y}N$, where N is the number of voters in the economy, and $\bar{y}$ is the mean wealth. Let the level of expenditure on the public good, $C(g)$, be such that $C'(g) > 0$ and $C(0) = 0$. If public expenditures are financed by proportional income taxed, and t is the tax rate, then $C(g) = t\bar{y}N$.

The ith individual votes for the level of public expenditure which maximizes his utility. We represent the utility as a function of the expenditure on the public good and on the private good, p_i. p_i is just his after tax wealth, $p_i = (1-t)y_i$. Thus she maximizes $U^i(g, (1 - \frac{C(g)}{\bar{y}N})y_i)$ with respect to g, and her utility is maximized when

$$\frac{U^i_g}{U^i_{p_i}} = C'(g)\frac{y_i}{\bar{y}N}. \tag{A.1}$$

Assuming that individuals differ only with respect to their endowment, we can trace out the demand for the public good as a function of y_i. In particular, if U is quasi-concave on g, then preferences are single peaked, and the majority voting equilibrium will be the level of demand of the individual with median wealth, provided that the demand for g is monotonic in y_i (see Fig. A.1).

However, the level of g voted for may not be monotonic in y. Since preferences are single peaked, there is still a majority voting equilibrium. But the "median voter" is not the individual with median income.

Alternatively, if U is not quasi-concave on g and the level of g voted for is not monotonic in y, the preferences are not either single peaked nor single-crossing. However, it is not hard to find non-convex production technologies for the public good, like those in Fig. A.2, under which the preferences of voters would be top monotonic, and yet, neither single peaked not single crossing.

A.2. Choosing between public and private provision

This application is based on Epple and Romano (1996). There are two goods, educational services, x, and the numeraire commodity, b. All individuals have the same strictly increasing, strictly quasi-concave, and twice continuously differentiable utility function $U(x, b)$. It is also assumed that:

Assumption A.1. Educational services are a normal (or superior) good.

Assumption A.2. For $x > 0$, $b > 0$, $\bar{x}$ and $\bar{b} \geqslant 0$, $U(x, b) > U(0, \bar{b})$ and $U(x, b) > U(\bar{x}, 0)$.

Households differ in endowed income (i.e. numeraire commodity), y. The p.d.f. and c.d.f. of household income are denoted $f(y)$ and $F(y)$, respectively, with support $[\underline{y}, \bar{y}] \subseteq [0, \infty)$. We assume that $f(y)$ is continuous and positive over its support. We normalize the number of households to one and denote aggregate income by $Y = \int_{\underline{y}}^{\bar{y}} yf(y)\,dy$, which is also then equal to mean income.

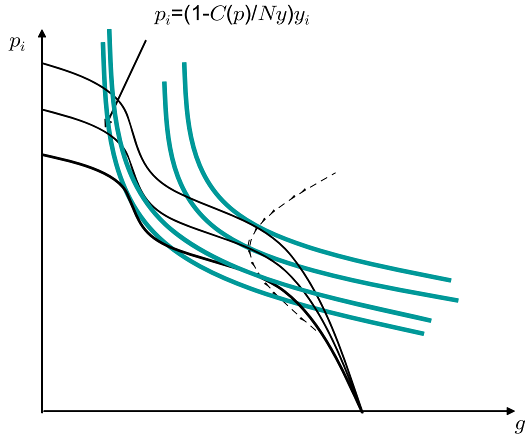

Fig. A.2. Demand for the public good as a function of y_i with a non-convex technology.

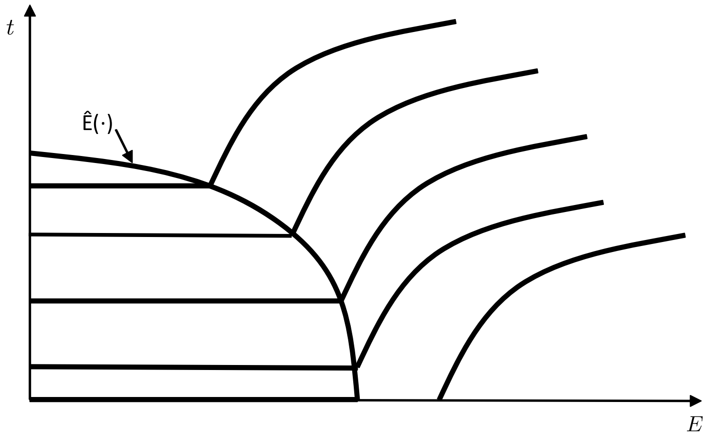

Fig. A.3. A typical indifference map in the (E, t) plane.

One unit of publicly provided educational services is produced with one unit of numeraire and all consumers of public school services obtain the same level of education services. Public school inputs are financed by a proportional tax, t, on income:

$$tY = NE,$$

where N is the number of households using public schools, and E is per household public school services. The level of public school expenditure is determined by majority voting of all households, whether or not they utilize public schools.

Private school services are provided by price-taking suppliers. The cost per unit of educational services provided by private schools is one unit of the numeraire. A household consuming private school services can choose as many units as it desires at price one per unit. A household can consume either public of private school services but not both.

A household that consumes private school services chooses x to maximize $U(x, b)$ subject to the budget constraint $y(1-t) = x + b$. Let $v(y(1-t))$ be the indirect utility function of a household with income y that chooses private schooling. The preferred tax rate for an agent choosing private services is $t = 0$.

A household with income y choosing public schooling obtains utility:

$$U\big(E, y(1-t)\big).$$

Let $E(t^*(y))$ be educational expenditure per household at the preferred tax rate for an agent choosing public services.

Hence, the induced utility function of a household with income y that can choose between public and private alternatives is

$$V\big(E, y(1-t)\big) = \max\big\{v\big(y(1-t)\big), U\big(E, y(1-t)\big)\big\}. \tag{A.2}$$

Let $\widehat{E}(y(1-t))$ be the locus of (E, t) pairs along which household y is indifferent between public and private school. A typical indifference map in the (E, t) plane is illustrated in Fig. A.3.

Let $E^*(t)$ be educational expenditure per household for those attending public school when all households make utility-maximizing choices. Note that $E^*(t)$ need not be everywhere increasing.

We now denote the slope of an indifference curve of $U(E, y(1-t))$ in the (E, t) plane be denoted by $M(E, y, t)$. Hence,

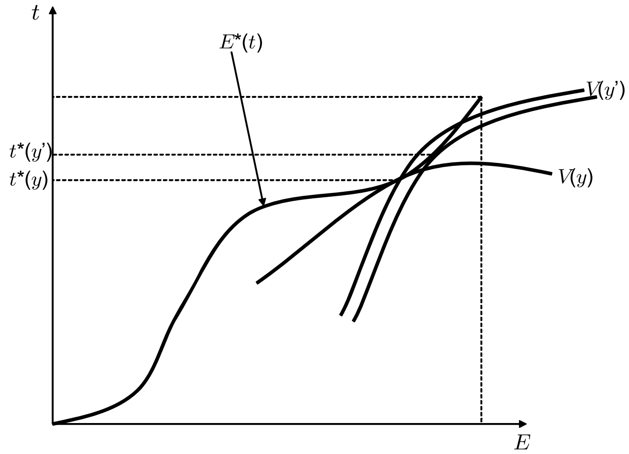

Fig. A.4. A utility function satisfying the single crossing condition SDI.

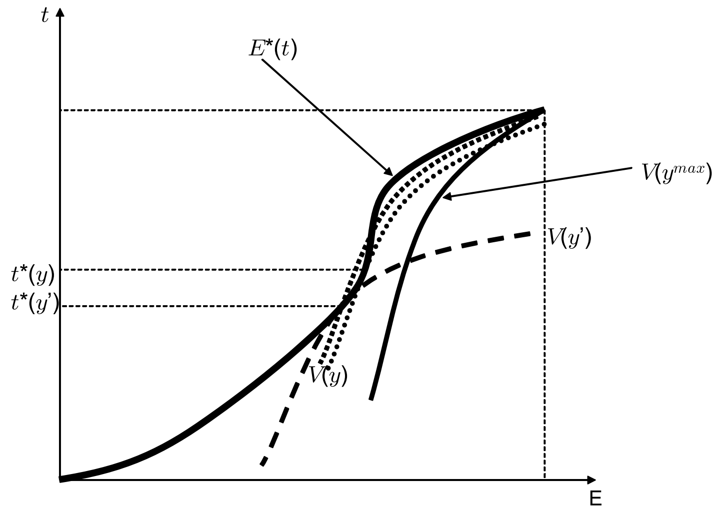

Fig. A.5. A utility function satisfying the SRI condition.

$$M(E, y, t) = \frac{U_1(E, y(1-t))}{yU_2(E, y(1-t))}.$$

Epple and Romano (1996) assume that for all y the slope of the $U(E, y(1-t))$ function in the (E, t) plane is monotonic in y. In particular, they assume that one of the following alternatives holds:

Assumption A3 or Slope Declining in Income (SDI). $\frac{\partial M(E,y,t)}{\partial y} \leqslant 0$ for all y.

Assumption A4 or Slope Rising in Income (SRI). $\frac{\partial M(E,y,t)}{\partial y} \geqslant 0$ for all y.

SRI results if the income elasticity of the implied demand for public education exceeds the (absolute value of the) price elasticity of the same, and SDI results is the reverse holds. Epple and Romano (1996) show that if the utility function satisfies SDI, then the single crossing condition is satisfied. Therefore, when SDI holds, a majority voting equilibrium exist, and the median voter is decisive. From Theorem 1, we know that any preference profile satisfying the single crossing condition is top monotonic (and since for this particular example essentially all agents have one maximal element, also peak monotonic).

We now illustrate how to check that if the utility function satisfies the single crossing condition SDI, then the preference profile is peak monotonic. Let $y > y'$ be two households (see Fig. A.4).

Suppose that either both households, y and y', choose public provision or household y chooses private provision and household y' chooses public provision. Then household y is such that either $t^*(y) < t^*(y')$ or $0 < t^*(y')$, and it is clear from the picture that household y prefers $t^*(y')$ to any $t^* > t^*(y')$. Household y' is such that either $t^*(y) < t^*(y')$, and it is clear from the picture that household y' prefers $t^*(y)$ to any $t^* < t^*(y)$ or $0 < t^*(y')$, and condition (2) in the definition of peak monotonicity is vacuously satisfied. Finally, since $\widehat{E}(y) > \widehat{E}(y')$ it cannot be the case that household y chooses public provision and household y' chooses private provision.

We finally present a graphical example (see Fig. A.5) in which the utility function satisfies the SRI condition, the preference profile is not single crossing but it is still peak monotonic.[13] We also assume that the highest income individual prefers $t = 1$ to private provision. It is clear that all individuals prefer public to private provision. Let $y > y'$ be two households.

Household y is such that $t^*(y) > t^*(y')$, and it is clear from the picture that household y prefers $t^*(y')$ to any $t^* < t^*(y')$. Household y' is such that $t^*(y) > t^*(y')$, and it is clear from the picture that household y' prefers $t^*(y)$ to any $t^* > t^*(y)$.

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[13] Admittedly, we need the individual with highest income be willing to accept full expropriation instead of private provision. However, this assumption may reflect the situation in some European countries.